variation is now 360° instead of a comparatively small sector (*e. g.*, 30° in the disk figured by Titchener).

All the objections urged against the Delboeuf disks apply with equal force to the Kirschmann photometer.[3] The photometers are difficult to make, and soon get dirty. The samples of paper to be tested, even when accurately cut, are difficult to center and may be smudged in pasting. We have accordingly applied Martin's arrangement to the photometer. If the motor is rapid, it is not necessary to cut two sector openings in the disk; one opening of 180° will not flicker. Thus we cut in a white cardboard disk (diam. 26 cm.) one 180° sector of an annular ring of radii 9 and 6.5 cm. (The arc of the smaller radius need not be cut carefully, since it lies beneath the next disks.) A disk of the paper to be tested, coupled with a disk of the white cardboard (diam. 14 cm.), forms the concentric variable ring which lies inside the black (hole) and white (cardboard) ring. A small white cardboard disk (diam. 10 cm.) fills the center. These disks can all be cut on the disk-cutter with so little handling that they are not likely to get dirty before use. No pasting is required, and centering is exact. But one incomplete arc (instead of five) has to be cut. The fine adjustment can be added, as usual, if it is desired. For very light grays it is well to have a large disk with a 90° annular sector instead of the 180° sector.

It is necessary, in using the Kirschmann photometer, carefully to fix the position of the observer so that he shall look directly into the long black tube, since the wall reflects some little light. We have found that a large black box (18 by 18 by 36 in.), with a circular hole cut at one end, makes a dark chamber whose sides are not brought into the observer's field of regard by any ordinary accidental shift of position.

III. Urban's Tables and the Method of Constant Stimuli

By E. G. Boring.

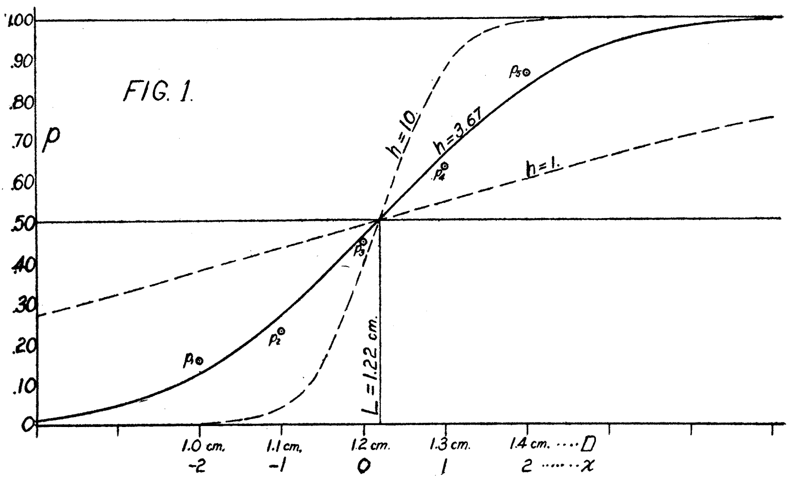
F. M. Urban's recent publications on the psychophysical methods, and in particular his *Hilfstabellen,* have so revised the procedure of the method of constant stimuli that the account in Titchener's *Quantitative Manual* is no longer adequate. In the Cornell Laboratory we have found it necessary to supplement the text of the *Manual* by individual instruction in the use of Urban's tables. The student cannot ordiarily be sent directly to the original articles, for the mathematics (and the German!) are usually beyond him. We propose, therefore, to print an elementary account of the method of constant stimuli in its present form. We shall use little mathematics. The instructor or student will, however, find in the notes at the end of this paper an indication of sources.

§ 1. The Psychometric Function

If for every member of a series of stimuli it is possible to give the one or the other of two judgments, and if it be found that the frequency with which the one judgment is given depends upon the value

[3] Titchener, *op. cit.*, 35ff.

of the stimulus and increases as we pass successively from one stimulus in the series to the next, we may say that the frequency, and hence the probability, of the occurrence of the given judgment is a function of the value of the stimulus. This function is called the *psychometric function.* Thus, if we have a series of separations of the aesthesiometer points, and if we find that the judgment 'two' is given almost not at all at the one end of the series, and almost universally at the other end, and increases in frequency between the two extremes, then a statement of the frequencies of these judgments for the successive stimulus-values is a statement of a psychometric function. The exact form of this function cannot be stated in advance and has never been determined even approximately for most experimental conditions. We know in general, however, that the series of frequencies increases continuously, although not at a constant rate, and that even for extreme stimuli there occur, though very rarely, judgments of the kind not characteristic of the particular extreme. In other words, if we take enough cases in determining the two-point limen, we shall never get quite as low as 0%, no matter how small the separation, nor quite as high as 100%, no matter how great the separation.



There is a mathematical function which satisfies these conditions, and which we might expect in general to apply in such a case; it is a form of the probability curve known as the Φ (γ) [phi function of gamma]. Such a curve is shown as a solid line in Fig. 1, which represents the frequency of the judgment 'two' as given for successive separations of the aesthesiometer (expressed in cm.). Theoretically the curve never quite reaches 0% or 100%, although these values may be obtained in an actual experiment based upon a relatively small number of cases. The presumption in favor of this form of curve lies in the fact that it expresses approximately the frequencies of any measure dependent entirely on chance. If, for instance, we measure the heights of a great many college men, and then plot successively the *per cent.* under 150 cm. in height, the *per cent.* under

151 cm., and so on by cm. up to (say) 190 cm., we get approximately this curve of the Φ (γ). The presumption that this curve would be a psychometric function has been borne out in practice so far as experiment has gone. Accordingly we are justified in making it the basis of our method.

It is clear from Fig. 1 that the theoretical curve or psychometric function is symmetrical about its middle point at 50%. This point is taken as the limen; that is to say, the limen is defined as that value of stimulus for which the probability of the judgment 'two' equals the probability of the judgment 'one.' The fixing of the liminal point, *L*, does not, however, determine the whole curve. Any number of Φ (γ)-curves can be drawn through this point, every one depending on the particular measure of precision, *h*, which attaches to it. The greater the value of *h*, the steeper the curve. For the solid curve of Fig. 1, $h = 3.67$. The steeper dotted curve is for $h = 10$; the flatter dotted curve is for $h=1$. If $h=0$ the curve would become a horizontal line; the limen would be indeterminate, and its precision zero.

§ 2. Selection of Stimulus-Values

The values of the stimulus must be selected on the basis of preliminary experiments so as to meet several requirements.

(1) The extreme stimuli must not give frequencies too close to 0 or to 100%. We are seeking to determine the point of the psychometric function which corresponds to 50%. Thus the more remote a given frequency is from 50%, the less weight can it be given as an index of the 50%-point. If we actually obtain 0% or 100% (results which violate the Φ (γ)-hypothesis), we are obliged, in computing the limen, to give them a weight of zero; that is to say, the terms disappear and the experimental work with those stimuli is entirely wasted.

(2) The stimuli should turn out to be grouped approximately symmetrically about the limen. Otherwise, the one judgment would be given oftener than the other, and we should run the risk of errors of habituation or expectation.

(3) It might appear from (1) that the best choice of stimuli would be a set in the immediate neighborhood of the limen. We cannot, however, push this argument too far. In the first place, we find that stimuli near the limen most frequently give rise to difficult judgments, and that the usually easy judgments of the more extreme stimuli exert a steadying effect upon the observer. What the extreme values lose in mathematical significance, then, is made up by their effect upon the attitude of the observer. In the second place, we must remember that, even should we wish to take all our stimuli close to the limen, we cannot do so because we cannot prophesy exactly where the limen will be situated. The closer together our stimuli are taken, the more easily will a slight divergence of the results from the anticipated frequencies render the distribution of the stimuli about the limen asymmetrical, and thus violate condition (2).

(4) The stimuli should be separated by equal intervals. The symmetrical distribution of (2) already implies this condition. The use of Urban's tables necessitates it (for the tables consist of calculated values which are based on this assumption).

(5) The number of stimuli chosen should be from five to seven. Experience shows that five is sufficient.

To satisfy the foregoing conditions we must determine by preliminary experiments a stimulus which gives a frequency between 10 and

20%, and another stimulus which gives between 80 and 90%. Then we must choose three other values which occur at equidistant intervals between these two extremes. We cannot afford, however, to employ values which the apparatus in use does not readily furnish. With the aesthesiometer we must use even scale-divisions. It may, therefore, be necessary to take six or seven stimuli instead of five, or to select a value slightly above or below one or both of the predetermined extremes. Even if this change destroys the symmetry of the stimuli about the expected position of the limen, it may be necessary. If the student is in doubt what values to select, he must consult the instructor.

We must never consider the choice of stimuli as final until the entire experiment is completed. After making the tentative choice described above, the values should be tried out in ten series. If these series indicate that the stimuli are likely to give frequencies which satisfy the necessary conditions, we may proceed with the method proper; but we must keep a watchful eye upon the results. If we find as the method progresses that our values were not wisely chosen, we must be willing to consider all our work thus far as preliminary, to choose new values, and to begin afresh. In practice, however, it seldom occurs that the indication of the preliminary series is refuted by the subsequent observations.

§ 3. Experimental Procedure

The directions in the *Student's Manual,* 103f., may be followed exactly. One hundred complete series are required. If Urban's tables are to be used, this number cannot be increased, since the tables are made out for even percentages only, and a greater number of series would give fractional percentages.

§ 4. Problem

The problem of this method may be illustrated by typical data, obtained by an undergraduate pair. In the experiment, five stimuli (D) of 1.0, 1.1, 1.2, 1.3, and 1.4 cm. gave frequencies (p) of 16, 23, 45, 63, and 86% respectively. (See columns 1 and 3 of the Table.) These actual values are plotted as separate points (p_1, p_2, *etc.*) in Fig. 1. Since the observed frequencies fit the $\Phi\ (\gamma)$-function approximately, but not exactly, we have to determine the particular $\Phi\ (\gamma)$-curve which best fits the results. The solution involves three principles. (*a*) We avail ourselves of the known properties of our hypothetical psychometric function by making use of a table which gives the relation between the values of p (ordinate) and γ (abscissa). (*b*) We must apply different weights to the different frequencies according as they are near to, or remote from, the critical 50%-point which determines the limen. (*c*) We must compute for these weighted values the *most probable* $\Phi\ (\gamma)$-curve by the method of least squares.

§ 5. The $\Phi\ (\gamma)$-Hypothesis

Let δ represent the (unknown) distance from the limen to any stimulus. Subliminal stimuli will then have negative values of δ; supraliminal stimuli, positive values. The actual values of δ vary inversely with the unit of measurement used. The smaller the unit, the larger is the number which states a given δ. The measure of precision, h, however, varies directly with the unit of measurement, so

TABLE

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
D	x	p	P	γP	xP	x^2P	$x\gamma P$	γ	s	sP	xsP	δt	γt	pt
1.0	-2	.16	.6921	-.4866	-1.3842	2.7683	.9732	-.7031	-2.2969	-1.5897	3.1794	-.22	-.807	.127
1.1	-1	.23	.8179	-.4273	-.8179	.8179	.4273	-.5224	-1.4776	-1.2085	1.2085	-.12	-.440	.267
1.2	0	.45	.9943	-.0883	0	0	0	-.0888	-.9112	-.9060	0	-.02	-.073	.459
1.3	1	.63	.9607	.2254	.9607	.9607	.2254	.2346	-.2346	-.2254	-.2254	.08	.294	.661
1.4	2	.86	.6463	.4937	1.2927	2.5853	.9875	.7639	.2361	.1526	.3052	.18	.661	.825
......	......	......	4.1113	-.2831	.0513	7.1322	2.6134	....	....	-3.7770	4.4677	....	....	......

$$h^1 = \frac{(4.1113)\ (2.6134) + (.2831)\ (.0513)}{(4.1113)\ (7.1322) - (.0513)\ (.0513)} = .3670$$

$$L^1 = \frac{(.0513)\ (.3670) + .2831}{(4,1113)\ (.3670)} = .2001$$

$$h = \frac{.3670}{.1} = 3.670$$

$$L = 1.2 + (.1)\ (.2001) = 1.22 \text{ cm,}$$

Check (addition):

$-3.7770 = .0513 - 4.1113 + .2831$
$= -3.7769$

$4.4677 = 7.1322 - .0513 - 2.6134$
$= 4.4675$

Check (solution):

$(7.1322)\ (.3670) - (.0513)\ (.3670)\ (.2001) = 2.6134$
$2.6137 = 2.6134$

that the precision is also always dependent upon the unit. Thus it happens that the product $h\delta$ is independent of the particular system of units used. This product is called γ.

$$\gamma = h\delta$$

γ depends on the frequency, p; and Fechner's Fundamental Table (*Student's Manual,* 99) gives the value of γ for each value of p, thus determining the general properties of the curve, although leaving the form to be finally determined in any particular case by the system of units and the value of h.

It is evident that the curve is completely determined by any two points. For suppose that two separations, D_1 and D_2, give frequencies, p_1 and p_2, and that we find from a table the values of γ_1 and γ_2 corresponding to p_1 and p_2. Then by definition of γ:

$$\gamma_1 = h\delta_1, \text{ and } \gamma_2 = h\delta_2.$$

But, if L is the limen, $\delta_1 = D_1 - L$ and $\delta_2 = D_2 - L$. Thus:

$$\gamma_1 = h\,(D_1 - L)$$
$$\gamma_2 = h\,(D_2 - L)$$

which can be solved simultaneously for the values L and h.

Since in our actual case the five frequencies do not fall exactly on the theoretical line, we should find that every pair would give us slightly different values of L and h. Thus, since our results are slightly inconsistent with our hypothesis, we must presently find the ***most probable*** **values of** L **and** h**, under our hypothesis, by the** ***method of least squares;*** but first we must *weight* our determinations.

§ 6. Weighting

The necessity for some sort of weighting is suggested by inspection of the curve of Fig. 1. The point p_3 (45%) fixes the position of L with much greater definiteness than does the point p_5 (86%). A change of 1% in p_3 would not shift L nearly so far to the one side or the other as an equal change in p_5 would tend to do; for at p_5 the abscissa-change for a unit-change in ordinate is relatively large. The exact values of these weights have been computed by Urban, and can be found from a table. See the column for P in Urban's tables. It will be observed that $P = 1$ for 50% and $P = 0$ for 100%. Such a relation was to be expected. The observed 50% must exert maximal influence upon the determination of the most probable 50%-point. The observation 100% contradicts the hypothesis, and can have no effect at all upon the results.

If we are now to determine the most probable values of L and h, we must write the full set of equations, multiply every equation through by its weight, P, and then apply to the weighted equations the method of least squares. These weighted equations are:

$$\gamma_1 = h(D_1 - L) \text{ with the weight } P_1$$
$$\gamma_2 = h(D_2 - L) \text{ with the weight } P_2$$
$$\gamma_3 = h(D_3 - L) \text{ with the weight } P_3$$

. . . . . .

§ 7. The Method of Least Squares

We wish to obtain the $\Phi(\gamma)$-curve which represents our observed percentages with the greatest degree of probability. Such a curve will be, as we learn from the theory of probabilities, the curve for which the sum of the squares of the deviations of observed percentages

from theoretical percentages represented by the curve shall be a minimum. It may be determined by the method of least squares. This method results in the formation of two *normal equations,* which involve the known values of D, P, and γ, and the unknown values of L and h. By solving the equations simultaneously for L and h, the limen is determined. Substituting the more general term, x, for the specific term, D, these normal equations are:

$$[x^2P] \,.\, h - [xP] \,.\, L.h = [x\gamma P] \quad . \quad . \quad . \quad (1)$$
$$-[xP] \,.\, h + [P] \,.\, L.h = -[\gamma P] \quad . \quad . \quad . \quad (2)$$

where the squared bracket indicates the sum of all the values enclosed, computed separately for every one of the stimuli used.

We might now look up in tables the values of P and γ corresponding to every p (*cf.* columns 3, 9, and 2 of our Table), and compute from these figures the values γP, DP, D^2P, and $D\gamma P$ (*cf.* columns 5, 6, 7, and 8) for every one of the five values of D. Since four-place numbers would have to be multiplied together, the work would be laborious and the chance for error great. Fortunately the publication of Urban's tables makes these multiplications unnecessary.

§ 8. Solution of the Problem with the Use of Urban's Tables

Our problem, graphically represented in Fig. 1, is to fit the best curve to a given set of observed points. If we arbitrarily alter the units in which the stimuli are measured, that is to say, the abscissa-scale, we do not change the procedure. L and h come out in terms of the new system of units, and can be changed back again to the old system, or indeed to any other system, as one may desire. In Fig. 1, for example, we may substitute for the values of D, 1.0, 1.1, 1.2, 1.3, and 1.4 cm., an arbitrary system of x-units, *viz.,* -2, -1, 0, 1, and 2 respectively. The only necessary condition is that, since the values of D are equidistant, the values of x must also be equidistant.

In accordance with this principle, Urban has computed in his tables the products, upon which the sums of the normal equations are based, for a set of 15 equidistant x's, *viz.,* -7, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7. But before we proceed to the solution of our problem by the use of Urban's tables, two warnings are necessary.

Arrange your work systematically! Errors are easy to make and hard to find. The Table accompanying this article shows a form in which the principal computations may be kept. But the scratch-sheets on which the additions and multiplications are made should also be kept in order, until the limen is found and checked.

Be careful of your signs! Urban's tables are for positive values of x and γ. The same figures apply when either x or γ or both of them are negative, but the signs of some quantities are altered and of others not. The columns must be summed up algebraically, and attention must be paid to the signs of the sums in substituting in the equations for h and L.

Turning now to our Table, we write the values of D in cm. in column 1. Since the D's are equidistant, we may select for column 2 any set of equidistant x's between —7 and +7. (We *might* use, for instance, the values 3, 4, 5, 6, 7.) We take the values from —2 to +2, because they are small numbers, because they include the simple multiplications by 1 and 0, and because they will give in the long run the fewest negative quantities.

If we write the percentages, p, in column 3, we are ready to fill in columns 4 to 8 from Urban's tables. It is most convenient to fill in a line at a time. Let us neglect, for the moment, the signs. In the first line, $p = .16$. Urban's tables do not read below $p = .5$, because they are symmetrical. The values for .16 are thus the same as those for .84 (1.00—.16 = .84). The values of P and γP for $p = .84$ can be read directly from the second and third columns. Since $x = 2$ (temporarily neglecting the sign), xP becomes $2P$, x^2P becomes 2^2P, and $x\gamma P$ becomes $2\gamma P$. Urban's columns for $2P$, 2^2P, and $2\gamma P$, at $p = .84$, thus give the remaining values in our first line. In the second line $p = .23$, which is the same as $p = .77$. P and γP are found as before. Since $x = 1$, xP and x^2P are the same as P, and $x\gamma P$ is the same as γP. In the third line the last three values become zero, since $x = 0$. In the last two lines, p is greater than .5 and is found directly in the table.

But we must remember our signs! P, the weight, is never negative; a negative weight has no meaning; we cannot weight a thing less than zero. Hence all the values of column 4 are positive. Now we have seen that δ is negative when p is less than .5, for δ is the distance of any point from the limen and must be measured backwards for percentages less than 50%. Since $\gamma = h\delta$, and since P is always positive, γP, like δ, must be negative when p is less than .5. Thus the first three values of column 5 must be put down as negative. In column 6, xP is negative whenever x is negative. The next value, x^2P, is always positive, since neither x^2 nor P can be negative. Finally $x\gamma P$ depends upon both x and γ for its sign. If either x or γ is alone negative, then $x\gamma P$ is negative; but, if x and γ are both positive or both negative, then $x\gamma P$ is positive.

Columns 4 to 8 must be summed up *algebraically*.

If we let h^1 and L^1 stand for the constants of the psychometric function as expressed in the system of x-units (not in cm.), we can find their values from the normal equations, (1) and (2), of the method of least squares. It is simplest to find h^1 first, and then use h^1 in determining L^1. Solving the normal equations simultaneously, we get.

$$h^1 = \frac{[P]\ [x\gamma P] - [\gamma P]\ [xP]}{[P]\ [x^2P] - [xP]\ [xP]}$$

Then, substituting h^1 in equation (2),

$$L^1 = \frac{[xP]\ h^1 - [\gamma P]}{[P]\ h^1}$$

Substituting the sums in these equations (see Table), we find $h^1 = .3670$ and $L^1 = .2001$.

We have now to transform our results into cm. Let d be the number of units of D which correspond to a single unit of x. In this case, $d = 0.1$. Let m be the value of D which corresponds to $x = 0$. In this case, $m = 1.2$ cm. Then,

$$h = \frac{h^1}{d}, \text{ and } L = m + dL^1.$$

Substituting (see Table), $h = 3.760$ and $L = 1.22$ cm. (When $d = 1$, as would have been the case had the D's been expressed in mm., the relation simplifies: $h = h^1$ and $L = m + L^1$.)

§ 9. Checking

We have solved our problem, but we cannot yet be sure that our numerical results are correct. The chance of error is reduced by the use of Urban's tables; but still we may make a mistake in copying, in the determination of a sign, in adding algebraically, in substituting in the equations, or in solving them. If we are working exactly, we must check our results, even at the expense of a great deal of labor.

Checking the Sums. We can check all our work as far as the sums in columns 4 to 8 in the following manner. In column 9 write the values of γ, not forgetting the sign. These values can be obtained from the corrected Fechner table (see note to § 5). In column 10 are values of s, a sum which is arbitrarily defined as

$$s = x - 1 - \gamma \qquad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Attention must be paid to the signs of both x and γ in computing s. In column 11 multiply out the products, sP, and in column 12, xsP. Sum up columns 11 and 12.

Multiplying equation (3) by P, we get,

$$sP = xP - P - \gamma P \qquad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Summing up these terms,

$$[sP] = [xP] - [P] - [\gamma P] \qquad . \quad . \quad . \quad (5)$$

If we multiply (4) by x and then sum up the terms, we have,

$$[xsP] = [x^2P] - [xP] - [x\gamma P] \qquad . \quad . \quad . \quad (6)$$

Equations (5) and (6) contain in their right-hand members all the sums with which we are concerned. The left-hand members are the sums just found for the purpose of the check. The agreement between the two sides should be correct to three places. In our example we get (see Table)

$$-3.7770 = -3.7769$$

and

$$4.4677 = 4.4675.$$

If (5) fails to check and (6) checks, the error must be in $[sP]$, $[P]$, or $[\gamma P]$. If (6) fails to check and (5) checks, the error must be in $[xsP]$, $[x^2P]$, or $[x\gamma P]$. If both equations fail to check, there is a strong presumption that the error is in $[xP]$, which occurs in both equations.

Checking the Solution of the Equations. We found h^1 by solving the two normal equations simultaneously. We found L^1 by substituting h^1 in normal equation (2). We can check these solutions by substituting the values found for h^1 and L^1 in normal equation (1). In our particular example (see Table) we get

$$2.6137 = 2.6134.$$

Graphic Checking. The plotting on graph-paper of the theoretical curve given by L and h, and its comparison with the position of the points which represent the observed frequencies, constitutes a rough check, which may be substituted for the two foregoing checks if exact results are not required. If the curve as plotted appears to be representative of the observed points, it may be concluded that no gross errors have occurred.

To plot the curve, write the deviations of D from the determined limen (column 13 of Table): $\delta_t = D - L$. From δ_t calculate the corresponding theoretical values of γ: $\gamma_t - h\delta_t$ (column 14). Look up the theoretical percentages, p_t (column 15), in a table of the $\Phi(\gamma)$.

Plot p_t against D to give the theoretical curve. It must cut the 50% abscissa at the ordinate of the limen. Indicate the position of the observed percentages, p, by dots or small circles. The graph will have the form of Fig. 1. It is not necessary, however, to extend it beyond the extreme stimuli.

§ 10. THE METHOD OF CONSTANT STIMULUS DIFFERENCES

The application of the foregoing method to the problem of the determination of an upper and a lower DL requires little further exposition. The lower DL is obtained from the p's for the judgment 'less;' the upper DL from the p's for the judgment 'greater.' The 'equal' judgments are not considered separately.

In reporting such experiments it is customary to give the value of the *interval of uncertainty*, which is the difference between the two limens.

$$\textit{Interval of uncertainty} = L_U - L_L.$$

The interval of uncertainty is not necessarily a region of subjective equality; for, if the curve of 'equal'-judgments is skewed, the greatest frequency of the equality-judgment may occur outside of the interval of uncertainty.

There are four possible definitions of the *point of subjective equality.*

(a) Urban defines it as the point at which *the probability of the judgment 'greater' equals the probability of the judgment 'less,' i.e.,* the point at which the two psychometric functions intersect. This point is given by the value $\dfrac{h_U L_U + h_L L_L}{h_U + h_L}$.

(b) Frequently $h_U = h_L$ approximately. Then the formula becomes $\dfrac{L_U + L_L}{2}$. On the assumption that this relation holds, the point of subjective equality may be taken as *the average of the two limens.*

(c) Considering the psychometric function for the 'equal'-judgments instead of the psychometric functions for 'greater' and 'less,' we may define subjective equality as *the average equality-judgment.* If p_1, p_2, p_3, *etc.*, are the percentages of 'equal'-judgments for the stimuli x_1, x_2, x_3, *etc.*, then the point of subjective equality would be given by $\dfrac{p_1x_1 + p_2x_2 + p_3x_3 + \dots\dots}{p_1 + p_2 + p_3 + \dots\dots}$.

(d) Finally, the point of subjective equality may be taken as *the most probable value of the equality-judgment,* that is to say, the maximal point of the curve of 'equals.' This point is found by taking the three maximal frequencies, p_a, p_b, and p_c, corresponding respectively to the stimuli x_a, x_b, and x_c. If the three values of x are equidistant, the stimulus corresponding to the maximal point is $x_b + \dfrac{(p_a - p_c)(x_c - x_b)}{2(p_a + p_c - 2p_b)}$

In a symmetrical distribution the four measures may coincide. Usually their separate calculation and comparison is of interest.

§ 11. Notes

For the history of the method one should, of course, still read Titchener, *Quantitative Student's Manual,* 275ff. The discussion in the *Quantitative Student's Manual,* 92ff., still applies when Urban's tables cannot be used, provided that certain changes are made in the manner of choosing the stimulus-values (§ 2 of this paper), and that the necessary changes are made in the fundamental table (§ 5 and note) and in the table of weights (§ 6 and note). For Urban's account of the modified method, see *Die Praxis der Konstanzmethode,* Leipzig, 1912, 26pp. This pamphlet includes the 'short-cut' tables in correct form. The *Arch. f. d. ges. Psychol.,* 24, 1912, 236ff., contains the tables (with two mistakes) and a briefer indication of their application. On the method in general, see Urban in *Psychol. Rev.,* 17, 1910, 229ff. This article is based on the fuller accounts in *Arch.,* 15, 1909, 261ff.; 16, 1909, 168ff. S. W. Fernberger's monograph, *Psychol. Rev. Monog.,* No. 61, 1913, gives a clear and readily available account of the use of the method (without the 'short-cut' tables) and also of the method of checking.

Since Urban's notation differs from the older notation adopted by Titchener, we must make a choice at the outset; and, since Urban's symbols are those now current in mathematical texts, we shall select them. In comparing the following discussion with Titchener's *Manual* the student should therefore bear in mind that $\gamma = t$, $p = n$, and $P = w''$ (and, since w' is usually unity, and therefore $w = w'w'' = w''$, $P = w$ as a rule). If we let D still stand for the actual stimulus-values, then we can use Urban's x for the corresponding arbitrary values of the table. L will be the limen.

§ 1. The student must refer again to Titchener's discussion of the law of error, *Student's Manual,* 38ff.

It must be remembered that the use of the $\Phi(\gamma)$-hypothesis is not essential to the method of constant stimuli. On the solution of the problem by the arctan-hypothesis and by indifferent interpolation by Lagrange's formula, see Urban, *Arch.,* 15, 335ff.; 16, 205ff.; *Psychol. Rev.,* 17, 233ff., 257ff.

§ 2. On the adequacy of five stimuli to exact results, even when two limens (an upper and a lower DL) are to be computed, see Fernberger, *Amer. Jour. Psychol.,* 25, 1914, 121ff.; *Psychol. Rev.,* 21, 1914, 335ff.

The assumption of our discussion is that a range of frequencies from 15% to 85% is ideal. These values are arbitrary. They are, however, sufficiently removed from 50% to satisfy condition (3); and they are weighted in the computation by ⅔ (.6697), so that they still play an important part in determining the limen, as is required by (1).

The present method differs in several counts from Riecker's procedure in the experiment outlined in the *Student's Manual,* 92ff. (*a*) The inclusion of zero separation (one point) may be required under particular conditions, but there is no more reason for using it than there is for using any of the other values. *Vexirversuche* as controls should not be necessary with an observer trained against the stimulus-error. (*b*) The highest stimulus-value (6 Paris lines) should not have been used, for it gave a frequency of 100%. (*c*) The stimuli should have been equally spaced. (*d*) Too many stimuli were used. Five, or at most seven, would have been enough. (*e*) If we reduce the number of our stimuli, we cannot afford to throw out inversions;

and, besides, every observed frequency has a right to be counted to the extent that its particular weight allows.

§ 3. One does not ordinarily wish to take less than 100 series. The tables will work, of course, for any even factor of 100; *i.e.,* 50, 25, 20 series.

The question arises whether *O* should be told when a new series is begun. It appears at present that, in the most careful work, he should make every single judgment absolutely independently, without any reference to the rest of the experiment; *cf.* S. S. George, *Amer. Jour. Psychol.,* 28, 1917, 1ff. (especially 33ff.). Under George's conditions the announcement of the beginning of a new series would be undesirable.

§ 4. *Cf. Student's Manual,* 102f.

§ 5. Urban has shown that Fechner's table (*Student's Manual,* 99) contains slight errors (from .0001 to .0003) in 15 of its values. The corrected table, calculated from Bruns' table of the probability integral, is printed in *Arch.,* 16, 180; *Psychol. Rev.,* 17, 251; W. Brown, *The Essentials of Mental Measurement,* 1911, 134. The student should correct the table in his *Manual* from one of these sources, and change t to γ and n to p. Although this table is not used in the 'short-cut' method, it is necessary in checking the results (§ 9).

§ 6. Urban's values of the weights, *P,* differ radically from Müller's; *Student's Manual,* 101. In some cases they are twice as great. Urban uses a different formula. For the formula, see *Arch.,* 16, 181; *Psychol. Rev.,* 17, 252; *Konstanzmethode,* 17. The actual values of the weights are tabulated to three decimal places in *Arch.,* 16, 183; *Psychol. Rev.,* 17, 253; Brown, *op. cit.,* 135; and to four places (since they constitute the first column of the *Hilfstabellen*) in *Arch.,* 24, 240; *Konstanzmethode,* 20. Since the student must be provided with a set of the *Hilfstabellen* (see § 8, note), he should simply disregard Müller's table in the *Manual.*

In the equations, P $(=w)$ must affect both sides, if the equations are to hold. The corresponding formulae in the *Student's Manual,* 102, are misprinted; w should apply to the right-hand side as well as to the left.

The precaution in *Student's Manual,* 100, still holds. If different frequencies are based on different numbers of observations, then the equations must also be weighted in proportion to the number of observations taken for every one; but the method of constant stimuli in its usual form prescribes the same number of observations for every stimulus.

§ 8. The tables in the *Konstanzmethode,* 20f., are correct. Those in *Arch.,* 24, 240f., have only two errors in the column for 6^2 P, when $p = .89$ and .90. These values are rarely used. See *Arch.,* 25, Literaturber. 84, for corrections. The table is more easily read if ruled horizontally every ten lines. In the Cornell Laboratory we have a negative of the correct table, thus ruled, from which we take blueprints for the students. The blue-print costs about five cents and can be pasted in the *Manual.* When carefully made, it is quite as legible as the original. We can furnish blue-prints from our negative to any laboratories that may desire them.

The danger of error is minimized by using direct formulae for h^1 and L^1, instead of having the student solve the normal equations simultaneously. Some undergraduates do not remember how to solve simultaneous equations; many make mistakes. Urban gives equations of

similar form, *Arch.*, 16, 186. We have avoided the use of the symbol c ($=hL$), because its exclusion simplifies the discussion and because the mathematical significance of this product (c) is not apparent to the non-mathematical student.

§ 9. On the first check, see Fernberger, *Psychol. Rev. Monog.*, No. 61, 32ff.; Urban, *Konstanzmethode*, 24f. Fernberger's statement that the check should be exact applies to the long method. The discrepancy in the fourth decimal place in our example arises from the fact that the values in Urban's tables are given to four places, but are computed from five-place P's and γ's; whereas our values of sP and xsP are based on four-place P's and γ's.

The second check unfortunately involves a multiplication ($h\,[xP]$) which was made in finding L^1. The second term of the left-hand member is therefore best found by multiplying in the order $L^1 \times [xP] \times h^1$.

While the graphic check is rough, it has the great pedagogical advantage of showing the student diagrammatically just what the method has accomplished. It is very simply applied and is especially useful when the student's time is limited. We ordinarily use this check in the Cornell drill-course, whether or not the others are omitted.

Fechner's table may be used to change γ_t into p_t. Since the graphic check is rough, the table in the *Students Manual*, 99, can be used even if uncorrected. It is much more convenient, however, to employ a table that reads from γ to p, instead of from p to γ. Such a table is to be found in B. Kämpfe, *Philos. Stud.*, 9, 1893, 147ff. (not *Psychol. Stud.*, as Urban says); and in H. Bruns. *Wahrscheinlichkeitsrechnung und Kollecktivmasslehre*, 1906. It is from the Bruns table that Urban computed the corrections of the Fechner table. Both these tables give values of Φ (γ), which must be changed into p by dividing by 2, and adding to .5 when γ is positive, or subtracting from .5 when γ is negative. We have the Bruns table on two negatives in the Cornell Laboratory, and can furnish blue-prints of them. For the rough graphic check the Bruns table is unnecessarily accurate. Miss J. M. Gleason has prepared a mimeographed table which gives p directly for values of γ to two decimal places. Miss Gleason has deposited the stencil and a large number of copies of this table with the Cornell Laboratory, so that we can also furnish limited numbers of this table to those who may desire them.

It may be pointed out that, if the values of γ_t are computed accurately and the values of p_t found accurately by interpolation in the Bruns table, then the sum of the squares of the differences between p_t and the observed percentages, p, constitutes a measure of the degree with which the actual case fits the hypothesis.

§ 10. The formula of (a) is easily derived if we take x_e as the point of subjective equality, p_e as the percentage of the point of intersection (subjective equality), and γ_e from the table for p_e. Since γ_e is common to both psychometric functions, it can be expressed with respect to both; thus,

$$\gamma_e = \delta_e L h L = (x_e - L_L)\, h_L,$$
$$\gamma_e = -\, \delta_e U h U = (L_U - x_e)\, h_U.$$

Equate the two right-hand members and solve for x_e to get the formula. See Urban, *Arch.*, 16, 201.

The formula of (*d*) is found by assuming that a parabola may be used as the interpolated curve (a parabola is determined by three points), and finding the maximum by equating the first derivative to zero. For the derivation, see Urban, *The Application of Statistical Methods to the Problems of Psychophysics,* 1908, 124f. The formula given here has A—C in the numerator and is correct. It has been misprinted A + C in *Arch.,* 16, 187, and *Psychol. Rev.,* 17, 236ff.; but with this caution in mind, see those discussions.

For examples of asymmetrical psychometric functions, see *Arch.,* 16, 199ff.

MINOR STUDIES FROM THE PSYCHOLOGICAL LABORATORY OF CORNELL UNIVERSITY

Communicated by E. B. Titchener and H. P. Weld

XXXIV. Size *vs.* Intensity as a Determinant of Attention

By J. N. Curtis and W. S. Foster

This study is an attempt to compare the attention-compelling power of size and intensity in the case of Greek crosses. A standard cross, identical with that used by Meads [1], the arear of which was 56 sq. cm., was compared with two similar crosses whose areas were respectively 28 and 112 sq. cm.

The apparatus used was that of Meads, save that a pendulum tachistoscope was substituted for the spring tachistoscope. The average time of exposure was also the same (110 sigma), but the mean variation of this average was reduced from 8 to 3 sigma. A 40-watt Mazda lamp was used, and the standard cross had the intensity of 225° of light.

Preliminary experiments by the method of limits indicated that results as definite and constant as those in which 'form' and intensity were compared could not be obtained for size. To rule out, so far as possible, any influence of expectation, we turned, in the regular series, to the method of constant stimulus-differences. The experiments were arranged to compensate for the irregular influences of practice and fatigue, and to measure the error of space.

The observers were Dr. E. G. Boring, Dr. W. S. Foster, and Mr. F. L. Dimmick, all highly practised. All observers completed 200 series; 50 with each of the two comparison crosses in each of the two spatial positions, right and left of the standard. The period of observation was approximately an hour in length, and in general gave time for ten series. Rest-periods of three to five minutes were allowed twice during the hour. The observer was not told the number of steps in a series, nor did he get any indication of the point at which one series ended and another began.

The instructions, which were read at the beginning of every experimental hour, were: "At 'now' put your attention definitely upon the fixation-point. Two crosses of unequal size will be exposed. Judge which of them, if either, is the more *clear,* i. e., which one of them

[1] *Am. Jour. Psych.* xxvi., 1915, 150.